SEMICONFIGURATION SPACES OF PLANAR LINKAGES

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Abstract. This paper characterizes which subsets of $\mathbb{C}^n$ can be the set of positions of n points on a linkage in $\mathbb{C}$. For example, assuming compactness they are just compact semialgebraic sets. Noncompact configuration spaces are semialgebraics sets invariant under the Euclidean group, with compact quotient.

1. Linkages

Loosely speaking, a linkage is an ideal mechanical device consisting of a bunch of stiff rods sometimes attached at their ends by rotating joints. A planar realization is some way of placing this linkage in the plane. The configuration space for a linkage is the space of all such planar realizations, which can be determined by looking at all possible positions of the ends of the rods. Such configurations spaces have been studied for example in [5] and [4]. In this paper, we look at semiconfiguration spaces of linkages, where we look at all possible positions of only some of the points on the linkage. For example, what curve does a particular point on the linkage trace out? We give a complete description of possible semiconfiguration spaces in Theorem 1.1 below. For example, compact semiconfiguration spaces correspond exactly to compact semialgebraic sets.

Suppose L is a finite one dimensional simplicial complex, in other words, a finite set $\mathcal{V}(L)$ of vertices and a finite set $\mathcal{E}(L)$ of edges between certain pairs of vertices. An *abstract linkage* is a finite one dimensional simplicial complex L with a positive number $\ell(\overline{vw})$ assigned to each edge $\overline{vw}$, i.e., a function $\ell\colon \mathcal{E}(L) \to (0,\infty)$. A *planar realization* of an abstract linkage (L,ℓ) is a mapping $\varphi\colon \mathcal{V}(L) \to \mathbb{C}$ so that $|\varphi(v) - \varphi(w)| = \ell(\overline{vw})$ for all edges $\overline{vw}$.

We will often wish to fix some of the vertices of a linkage whenever we take a planar realization. So we say that a *planar linkage* $\mathcal{L}$ is a foursome (L,ℓ,V,μ) where (L,ℓ) is an abstract linkage, $V \subset \mathcal{V}(L)$ is a subset of its vertices, and $\mu\colon V \to \mathbb{C}$. So V is the set of fixed vertices and μ tells where to fix them. The configuration space of realizations is defined by:

$$\mathcal{C}(\mathcal{L}) = \left\{ \varphi\colon \mathcal{V}(L) \to \mathbb{C} \;\middle|\; \begin{array}{cl} \varphi(v) = \mu(v) & \text{if } \ v \in V \\ |\varphi(v) - \varphi(w)| = \ell(\overline{vw}) & \text{for all edges } \overline{vw} \in \mathcal{E}(\mathcal{L}) \end{array} \right\}$$

If $W \subset \mathcal{V}(L)$ is a collection of vertices, the semiconfiguration space is the set of restrictions of realizations to W,

$$\mathcal{SC}(\mathcal{L}, W) = \{\, \varphi\colon W \to \mathbb{C} \mid \text{there is a } \varphi' \in \mathcal{C}(\mathcal{L}) \text{ so that } \varphi = \varphi'|_W \,\}$$

If we order the elements of W as $w_1, w_2, \ldots, w_k$ then there is a natural identification of $\mathcal{SC}(\mathcal{L}, W)$ with a subset of $\mathbb{C}^k$ where φ is identified with the point $(\varphi(w_1), \ldots, \varphi(w_k))$. With this identification, we see that the semiconfiguration

space of a linkage is the projection of its configuration space to some coordinate subspace.

Note that $\mathcal{C}(\mathcal{L})$ is a real algebraic set inside $\mathbb{C}^n$ since it is given by polynomial equations of the form $z_i =$ a constant and $|z_i - z_j|^2 =$ a constant. So $\mathcal{SC}(\mathcal{L}, W)$ is the projection of a real algebraic set. By the Tarski-Seidenberg theorem [6], projections of real algebraic sets are semialgebraic sets. A *semialgebraic set* is a finite union of sets of the form

$$\{ x \in \mathbb{R}^n \mid p_i(x) = 0, q_j(x) \geq 0, \text{ and } r_k(x) > 0 \}$$

for collections of polynomials p_i, q_j and r_k. In other words it is the closure under the boolean operations of finite union, intersection, and complement, of the family sets of the form $p^{-1}([0, \infty))$, where p is a polynomial.

We are now ready to completely characterize semiconfiguration spaces. Up to linear maps, they are just compact semialgebraic sets cartesian product with $\mathbb{C}^m$. But we can be even more precise.

Let Euc(2) denote the group of Euclidean motions of $\mathbb{C}$. So a general element of Euc(2) is a map $z \mapsto \omega z + z_0$ or $\omega \overline{z} + z_0$ where $\omega \in \mathbb{C}$ satisfies $|\omega| = 1$. We say a subset $Z \subset \mathbb{C}^k$ is *virtually compact* if Z is either compact, or it is invariant under the diagonal action of Euc(2), with compact quotient.

Theorem 1.1. *Suppose $X \subset \mathbb{C}^n$.*

1. *The following are equivalent:*
   (a) *There is a linkage $\mathcal{L}$ and a $W \subset \mathcal{V}(\mathcal{L})$ so that $\mathcal{SC}(\mathcal{L}, W) = X$.*
   (b) *After perhaps permuting the coordinates, $X = Y_1 \times Y_2 \times \ldots \times Y_m$ where each $Y_i \subset \mathbb{C}^{k_i}$ is a virtually compact semialgebraic set.*
2. *Moreover, for connected linkages the following are equivalent:*
   (a) *There is a connected linkage $\mathcal{L}$ and a $W \subset \mathcal{V}(\mathcal{L})$ so that $\mathcal{SC}(\mathcal{L}, W) = X$.*
   (b) *X is a virtually compact semialgebraic set.*

As an extra bit of information, if the connected linkage $\mathcal{L}$ has any fixed vertices or if W is empty, then $\mathcal{SC}(\mathcal{L}, W)$ is compact. Otherwise, $\mathcal{SC}(\mathcal{L}, W)$ is invariant under the diagonal action of Euc(2)*, with compact quotient, but $\mathcal{SC}(\mathcal{L}, W)$ itself is noncompact. In the general case if $\mathcal{L}$ is perhaps disconnected, then $\mathcal{SC}(\mathcal{L}, W)$ is compact if and only if each component of $\mathcal{L}$ which contains points of W also contains a fixed vertex.*

Example 1.1. We will illustrate Theorem 1.1 with an example. Suppose that $X \subset \mathbb{C}^2$ is the diagonal, $X = \{ (z_1, z_2) \in \mathbb{C}^2 \mid z_1 = z_2 \}$. Then X is Euc(2) invariant and $X/$Euc(2) is a single point. Let $\mathcal{L}$ be the linkage with five vertices A, B, C, D, E and with edges $\overline{AB}, \overline{BC}, \overline{AC}$ of length 1 and edges $\overline{AD}, \overline{BD}, \overline{CD}, \overline{AE}, \overline{BE}, \overline{CE}$ of length $\sqrt{5}/4$. Let $W = \{D, E\}$. We do not fix any vertices of $\mathcal{L}$. Then in any realization of $\mathcal{L}$, the vertices A, B, C form an equilateral triangle and the vertices D and E both must lie at the barycenter. Consequently, $\mathcal{SC}(\mathcal{L}, W) = X$. Note that the configuration space $\mathcal{C}(\mathcal{L})$ is higher dimensional, since the triangle ABC can rotate around D and E. In fact $\mathcal{C}(\mathcal{L})$ is a single orbit of Euc(2) (with no isotropy).

We can further characterize semiconfiguration spaces according to the number of fixed vertices. For clarity, we restrict attention to the connected case. The corresponding statement for disconnected linkages follows from the observation that

the semiconfiguration space of the disjoint union of two linkages is the cartesian product of their semiconfiguration spaces, (see Lemma 3.3).

Theorem 1.2. *Let $Z \subset \mathbb{C}^n$ be a virtually compact semialgebraic set, and suppose that Z is invariant under the action of a subgroup G of* $\mathrm{Euc}(2)$*, with compact quotient. Then there is a connected linkage $\mathcal{L}$ and a $W \subset \mathcal{V}(\mathcal{L})$ so that $\mathcal{SC}(\mathcal{L}, W) = Z$ and so that:*

1. *If G is* $\mathrm{Euc}(2)$*, then $\mathcal{L}$ has no fixed vertices.*
2. *If G is conjugate to $O(2)$, then $\mathcal{L}$ has only one fixed vertex, and that vertex is fixed at the fixed point of G.*
3. *If G is an order two subgroup generated by a reflection, then $\mathcal{L}$ has only two fixed vertices, and these vertices are fixed at points on the fixed line of G.*
4. *Otherwise, $\mathcal{L}$ has only three fixed vertices.*

Moreover, we have a converse to this result. Suppose $\mathcal{L}$ is a connected linkage with m fixed vertices and $W \subset \mathcal{V}(\mathcal{L})$, then:

5. *If $m = 0$, then $\mathcal{SC}(\mathcal{L}, W)$ is a semialgebraic set invariant under the action of* $\mathrm{Euc}(2)$*, with compact quotient.*
6. *If $m = 1$, then $\mathcal{SC}(\mathcal{L}, W)$ is a compact semialgebraic set invariant under the action of a subgroup conjugate to $O(2)$.*
7. *If $m = 2$, then $\mathcal{SC}(\mathcal{L}, W)$ is a compact semialgebraic set invariant under the action of the order two subgroup generated by a reflection.*
8. *If $m > 0$, then $\mathcal{SC}(\mathcal{L}, W)$ is a compact semialgebraic set.*

2. Constructing polynomial quasifunctional linkages

A linkage $\mathcal{L}$ is *quasifunctional* for a map $f : \mathbb{C}^n \to \mathbb{C}^m$ if there are vertices $w_1, \dots, w_n$ and $v_1, \dots, v_m$ of $\mathcal{L}$ so that if $p : \mathcal{C}(\mathcal{L}) \to \mathbb{C}^m$ and $q : \mathcal{C}(\mathcal{L}) \to \mathbb{C}^n$ are the maps $p(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$ and $q(\varphi) = (\varphi(w_1), \dots, \varphi(w_n))$, then $p = f \circ q$. We call $w_1, \dots, w_n$ the input vertices and $v_1, \dots, v_m$ the output vertices. So in other words $\mathcal{L}$ is quasifunctional if the output vertices are f of the input vertices. The *domain* of a quasifunctional linkage is $q(\mathcal{C}(\mathcal{L}))$. In general, repetitions of input and output vertices are allowed. But for convenience, for all quasifunctional linkages in this paper, we will assume there are no repetitions, i.e., $v_i \neq v_j$ and $w_i \neq w_j$ if $i \neq j$.

In [5] or [4], quasifunctional linkages were constructed for any real polynomial map $\mathbb{C}^n \to \mathbb{C}^m$. In fact these linkages had some stronger properties, which we do not need in this paper. We will reproduce these constructions here, simplified when appropriate. Essentially, polynomial quasifunctional linkages were constructed in the nineteenth century, although Kapovich and Millson pointed out some necessary corrections to the old constructions. In particular, whenever there is a rectangle in a linkage, it should be rigidified by adding another edge joining the midpoints of two opposite edges. This will prevent certain degenerate realizations which destroy quasifunctionality. In the diagrams below, we represent this rigidifying edge by a gray line. The second correction made in [5] is in the Peaucellier inversor below, which we correct here by adding a simulated cable (see below). One could also add a simulated telescoping edge as was done in [5].

Theorem 2.1 (Kapovich and Millson)**.** *For any real polynomial function $f : \mathbb{C}^n \to \mathbb{C}^m$ and any compact $K \subset \mathbb{C}^n$ there is a quasifunctional linkage $\mathcal{L}$ for f whose domain contains K, and whose input and output vertices are all distinct.*

FIGURE 1. How to put a joint in the middle of an edge

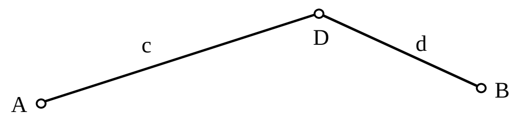

FIGURE 2. Simulating a cable or telescoping edge

The proof of Theorem 2.1 will occupy the rest this section. We now make some observations which reduce the proof of Theorem 2.1 to some special cases.

- The first observation is that by taking two such quasifunctional linkages and attaching the inputs of one to the outputs of the other, we obtain a quasifunctional linkage for the composition. Consequently, it suffices to find quasifunctional linkages for the elementary operations of addition, multiplication, and complex conjugation. Since $zw = (z+w)^2/4 - (z-w)^2/4$ we may replace multiplication by squaring and real scalar multiplication.
- The next observation is that there is a nontrivial quasifunctional linkage for the identity with domain all of $\mathbb{C}$. Recall Example 1.1 with semiconfiguration space the diagonal in $\mathbb{C}^2$. With D as input and E as output, this is a quasifunctional linkage for the identity with distinct input and output vertices. Consequently, by attaching these to the input and output vertices of any quasifunctional linkage, we may transform any quasifunctional linkage to one with distinct input and output vertices.

2.1. **Simulating interior joints, cables, and telescoping edges.** In our model of linkages, edges are connected only at their ends. Actual linkages used in real life might have a connection in the middle of an edge. This may be simulated as in Figure 1. In any realization, C must lie on the line segment from A to B. Thus when drawing linkages, it is allowable to draw a joint in the middle of an edge.

Although we will not use the following constructions in this paper in an essential way, we point out that using semiconfiguration spaces, we can also simulate other types of linkages. For example, suppose we want two vertices A and B connected by a cable, so the distance between them is constrained to be $\leq b$. More generally, suppose we wish to connect A and B by a telescoping edge, so the distance between them is constrained to be in the interval $[a, b]$. This can be simulated as in Figure 2. Since we are using semiconfiguration spaces, we can ignore the position of the vertex D. To simulate a cable, we take $c = d = b/2$. To simulate a telescoping edge with $0 < a < b$, we take $c = (a+b)/2$, $d = (b-a)/2$.

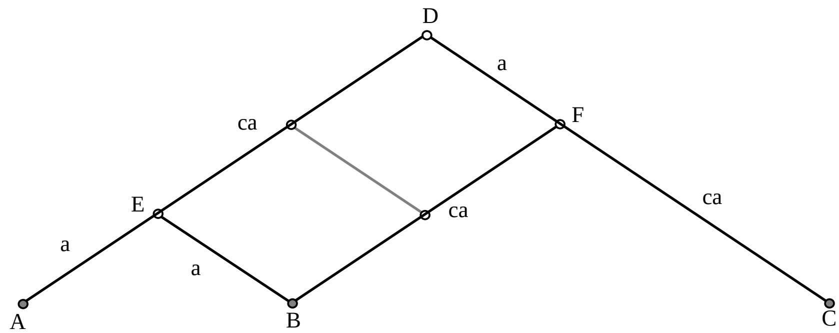

FIGURE 3. The Pantograph

2.2. **The useful pantograph.** We have reduced Theorem 2.1 to finding quasifunctional linkages for $(z, w) \mapsto (z + w)/2$, $z \mapsto \lambda z$ for λ real, $z \mapsto z^2$, and $z \mapsto \overline{z}$, all with domain containing an arbitrarily large compact set.

The first two functions can all be obtained from one type of linkage, the pantograph shown in Figure 3. It is a rigidified rectangle $DEBF$ with two extended sides. Because of the rigidification, in any realization the line $\overline{AD}$ is parallel to $\overline{BF}$ and the line $\overline{DC}$ is parallel to $\overline{EB}$. (Without the rigidifying gray edge, you would have realizations which folded half the figure about the line $\overline{DB}$ or $\overline{EF}$.)

Suppose $c = 1$ and the input vertices are A and C. Let the output vertex be B. Then $\mathcal{L}$ is quasifunctional for $(z, w) \mapsto (z + w)/2$. Its domain is all (z, w) with $|z - w| \leq 4a$ which can contain any compact set by choosing a big enough.

Next we will take the pantograph and find quasifunctional linkages for $z \mapsto \lambda z$, divided into three cases: $\lambda > 1$, $0 < \lambda < 1$, and $\lambda < 0$. In all cases the domain is an arbitrarily large ball.

- Suppose B is the input vertex, C is the output vertex, and A is fixed at 0. Then the linkage is quasifunctional for $z \mapsto (1 + c)z$ with domain $|z| \leq 2a$.
- Suppose C is the input vertex, B is the output vertex, and A is fixed at 0. Then the linkage is quasifunctional for $z \mapsto z/(1 + c)$ with domain $|z| \leq 4a$.
- Suppose A is the input vertex, C is the output vertex, and B is fixed at 0. Then the linkage is quasifunctional for $z \mapsto -cz$ with domain $|z| \leq 2a$.

2.3. **Inversion through a circle.** Before constructing the remaining quasifunctional linkages, we will find a quasifunctional linkage for inversion through a circle, $z \mapsto t^2 z/|z|^2$. This is shown in Figure 4. The linkage at the left is the full linkage, the one at the right just has the basics. The extra vertices and edges are only needed to rigidify $BDCE$ and eliminate some degenerate configurations which occur if B and C coincide.

We fix A at 0, set $t^2 = a^2 - b^2$, $c < b < a$, let the input vertex be D and the output be E. Let us see why $\mathcal{L}$ is quasifunctional for $z \mapsto t^2/\overline{z} = t^2 z/|z|^2$. If $\varphi \in \mathcal{C}(\mathcal{L})$ note that $\varphi(D)$ is a real multiple of $\varphi(E)$. This follows from the fact that the lines from $\varphi(A)$, $\varphi(D)$, and $\varphi(E)$ to the midpoint of the line segment $\varphi(B)\varphi(C)$ are all perpendicular to $\varphi(B)\varphi(C)$, hence $\varphi(A)$, $\varphi(D)$, and $\varphi(E)$ are

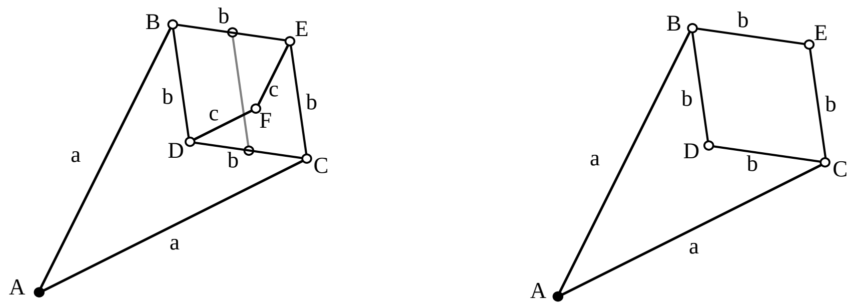

FIGURE 4. The Peaucellier Inversor

collinear. Solving triangles shows that if $s = |\varphi(B) - \varphi(C)|/2$, then

$$\begin{aligned} |\varphi(D)| &= \sqrt{a^2 - s^2} \pm \sqrt{b^2 - s^2} \\ |\varphi(E)| &= \sqrt{a^2 - s^2} \mp \sqrt{b^2 - s^2} \end{aligned}$$

from which we see that $|\varphi(D)|\,|\varphi(E)| = (a^2 - b^2)$. So $\mathcal{L}$ is quasifunctional for $z \mapsto t^2 z/|z|$.

To see the domain, note that $\sqrt{b^2 - c^2} \leq s \leq b$, so by the above,

$$\sqrt{a^2 - b^2 + c^2} - c \leq |\varphi(D)| \leq \sqrt{a^2 - b^2 + c^2} + c$$

So the domain is the annulus between the circles of radius $\sqrt{t^2 + c^2} \pm c$.

2.4. **How to square.** Now let us find a quasifunctional linkage for $z \mapsto z^2$ with domain containing $|z| \leq r$. Note that if $h(z) = t^2 z/|z|^2$ then

$$t^2 - th((h(t+z) + h(t-z))/2) = z^2$$

Suppose we take a quasifunctional linkage as above for h with $t = 4r$ and $c = 3r$, then the domain is $2r \leq |z| \leq 8r$. In particular, if $|z| \leq r$, then $t + z$, $t - z$, and $(h(t+z) + h(t-z))/2$ are all well within the domain. So by composition, we get a quasifunctional linkage for $z \mapsto z^2$ with domain containing $|z| \leq r$.

2.5. **Drawing a straight line.** So finally we are left with finding a quasifunctional linkage for complex conjugation. Our first step is to find a linkage so that some vertex is constrained to lie in a line segment. This linkage $\mathcal{L}$ is obtained by taking the input of a quasifunctional linkage for inversion through a circle and forcing this input to lie in a circle going through the origin. But when we invert a circle through the origin, we get a straight line. Now it is just a matter of translating and rotating it and rescaling, to make this line be any interval on the real axis. This linkage $\mathcal{L}$ is shown in Figure 5.

We fix C at 0 and fix B at $d\sqrt{-1}$. Let us first see what $\mathcal{SC}(\mathcal{L}, \{D\})$ is. If $\varphi \in \mathcal{C}(\mathcal{L})$, then we know from the analysis of Figure 4 that $\varphi(D)$ is in some annulus $0 < a \leq |z| \leq b$. On the other hand, because of the edge $\overline{BD}$ fixed at B, we must also have $|\varphi(D) - d\sqrt{-1}| = d$. Thus $\mathcal{SC}(\mathcal{L}, \{D\})$ is the intersection of the annulus $a \leq |z| \leq b$ with the circle $|z - d\sqrt{-1}| = d$. So as long as we choose d so $a < 2d < b$, we know this intersection is an arc of the circle. But then $\mathcal{SC}(\mathcal{L}, \{A\})$ is the inversion of this arc, which is a straight line segment.

By rescaling all side lengths by a fixed multiple, we may make this line segment as long as we wish. By translating and rotating the positions of the fixed points

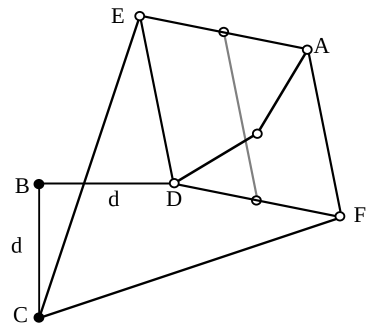

FIGURE 5. With B and C fixed, A will trace out a straight line

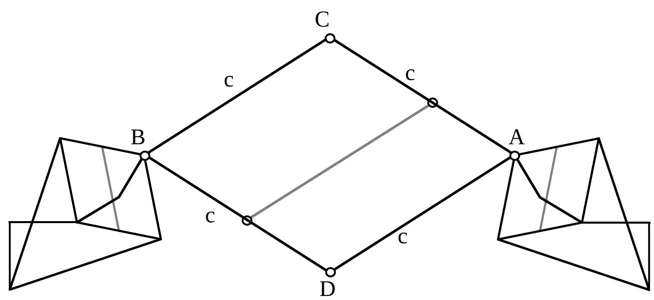

FIGURE 6. Complex Conjugation

C and B, we may translate and rotate this line segment to any other line segment with the same length. So we may construct such a linkage so that $\mathcal{SC}(\mathcal{L}, \{A\})$ is any line segment in $\mathbb{C}$ that we wish.

2.6. **Complex conjugation.** We are now ready to construct a quasifunctional linkage for complex conjugation. Our first step is to pick two linkages $\mathcal{L}'$ and $\mathcal{L}''$ as above with vertices A and B so that $\mathcal{SC}(\mathcal{L}', \{A\}) = [a, b]$ and $\mathcal{SC}(\mathcal{L}'', \{B\}) = [-b, -a]$ where $0 < a < b$. We then insert a rigidified square between A and B as shown in Figure 6.

Note then that if C is the input vertex and D is the output vertex, then $\mathcal{L}$ is quasifunctional for $z \mapsto \overline{z}$. If we choose a, b, and c so that $b - r > c > a + r$ then the domain will contain all z with $|z| \leq r$.

3. Proofs of Theorems 1.1 and 1.2

Now that we have finished proving Theorem 2.1, we may proceed with proving Theorem 1.1 and Theorem 1.2. But first we will need a few lemmas.

Recall that a continuous map $f : X \to Y$ is proper if $f^{-1}(K)$ is compact whenever $K \subset Y$ is compact. One useful property of proper maps is that if $A \subset X$, and X and Y are locally compact and Hausdorff, then $f(\mathrm{Cl}(A)) = \mathrm{Cl}(f(A))$. Here $\mathrm{Cl}(A)$ stands for the closure of A.

I imagine there is a more elementary proof of the following Lemma which does not use resolution of singularities, but in any case we have:

Lemma 3.1. *Suppose $X \subset \mathbb{R}^n$ is a semialgebraic set. Then there is a real algebraic set Y and a proper polynomial map $q\colon Y \to \mathbb{R}^n$ so that $q(Y)$ is the closure of X.*

Proof. First, we have an algebraic set $X' \subset \mathbb{R}^n \times \mathbb{R}^m$ so that if $p'\colon X' \to \mathbb{R}^n$ is induced by projection, then $p'(X') = X$. This is well known and easily illustrated by example. If $X = \{x \in \mathbb{R}^n \mid p(x) = 0, q(x) \geq 0, r(x) > 0\}$ for polynomials p, q, and r, then just let

$$X' = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^2 \mid p(x) = 0, y^2 = q(x), z^2 r(x) = 1\}$$

Next, let $X'' \subset \mathbb{R}^n \times \mathbb{RP}^m$ be the Zariski closure of X', i.e., the smallest real algebraic subset of $\mathbb{R}^n \times \mathbb{RP}^m$ which contains X'. ($\mathbb{RP}^m$ is real projective m space.) Let $p''\colon X'' \to \mathbb{R}^n$ be induced by projection. Note now that p'' is proper, since it is a restriction of the proper map $\mathbb{R}^n \times \mathbb{RP}^m \to \mathbb{R}^n$ to a closed subset.

It may be true that X'' is bigger than the closure of X', and $p''(X'')$ is bigger than the closure of X. So we must deal with this possibility. [1]

Let S be the singular set of X' and let $T = X'' - X' = X'' \cap (\mathbb{R}^n \times \mathbb{RP}^{m-1})$. By resolution of singularities (c.f., [3] or [2]), we know that there is a proper map $p'''\colon X''' \to X''$ so that X''' is nonsingular and $(p''')^{-1}(S \cup T)$ is a union of divisors with normal crossings. In particular, $(p''')^{-1}(X' - S)$ is dense in X''', so $p''p'''(X''')$ is the closure of $p'(X' - S)$.

But by induction on dimension, there is an algebraic set Y' and a proper polynomial map $q'\colon Y' \to \mathbb{R}^n$ so that $q'(Y')$ is the closure of $p'(S)$. Letting Y be the disjoint union $X''' \cup Y'$ and letting q be $p''' \cup q'$, we are done. □

Next we prove an important special case of Theorem 1.1.

Lemma 3.2. *Suppose $X \subset \mathbb{C}^n$ is a compact semialgebraic set. Then there is a linkage $\mathcal{L}$ and a $W \subset \mathcal{V}(\mathcal{L})$ so that $\mathcal{SC}(\mathcal{L}, W) = X$.*

Proof. By Lemma 3.1, there is a real proper polynomial map $q\colon Y \to \mathbb{C}^n$ where Y is a real algebraic set and $q(Y) = X$. By properness of q, we know that Y must be compact. By replacing Y with the graph of q, we may as well suppose that $Y \subset \mathbb{C}^n \times \mathbb{C}^m$ for some m, so that q is induced by projection $\mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$. Let $p\colon \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}$ be a real polynomial so that $Y = p^{-1}(0)$.

By Theorem 2.1, there is a quasifunctional linkage $\mathcal{L}'$ for p with distinct input and output vertices and whose domain contains Y. Construct a linkage $\mathcal{L}$ by taking $\mathcal{L}'$ and fixing its output vertex to 0. Let W be the set of the first n input vertices and let U be the set of all input vertices. Since the output vertex of $\mathcal{L}'$ is fixed to 0 and $\mathcal{L}'$ is quasifunctional for p we know that the input vertices must always lie on $Y = p^{-1}(0)$. So $\mathcal{SC}(\mathcal{L}, U) = Y$. Consequently, $\mathcal{SC}(\mathcal{L}, W)$ is the projection of Y to $\mathbb{C}^n$ which is X. □

Lemma 3.3. *Suppose $\mathcal{L}$ is the disjoint union of two linkages $\mathcal{L}'$ and $\mathcal{L}''$, and $W \subset \mathcal{V}(\mathcal{L})$. Then after reordering W so that the vertices in $W \cap \mathcal{V}(\mathcal{L}')$ come first, we have*

$$\mathcal{SC}(\mathcal{L}, W) = \mathcal{SC}(\mathcal{L}', W \cap \mathcal{V}(\mathcal{L}')) \times \mathcal{SC}(\mathcal{L}'', W \cap \mathcal{V}(\mathcal{L}''))$$

[1] For an example where this occurs, consider the case

$$\begin{aligned} X &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 < zy^2\} \\ X' &= \{(x, y, z, w) \in \mathbb{R}^4 \mid w^2(zy^2 - x^2) = 1\} \end{aligned}$$

One can show that $X'' = \{(x, y, z, [u : v]) \in \mathbb{R}^3 \times \mathbb{RP}^1 \mid u^2(zy^2 - x^2) = v^2\}$. But then points $(0, 0, z, [1 : 0])$ are in X'' but not $\mathrm{Cl}(X')$ if $z < 0$.

Proof. Any planar realization of $\mathcal{L}$ is a pair of realizations of $\mathcal{L}'$ and $\mathcal{L}''$, and vice versa. $\square$

Lemma 3.4. *Suppose $Z \subset \mathbb{C}^k$ is invariant under the diagonal action of* $\mathrm{Euc}(2)$*. Let $Z_0 = Z \cap (\mathbb{C}^{k-1} \times 0)$ be the points in Z with last coordinate 0. Then:*

1. *Z is virtually compact if and only if Z_0 is compact.*
2. *Z_0 is invariant under the diagonal action of $O(2)$.*
3. *$Z_0/O(2)$ is homeomorphic to $Z/\mathrm{Euc}(2)$.*
4. *If $Y \subset \mathbb{C}^k$ is invariant under the diagonal action of* $\mathrm{Euc}(2)$ *and $Z_0 = Y \cap (\mathbb{C}^{k-1} \times 0)$, then $Y = Z$.*

Proof. If $z \in Z_0$ and $\beta \in O(2)$ then $\beta(Z) \in Z$ and $\beta(z) \in \mathbb{C}^{k-1} \times 0$, so $\beta(z) \in Z_0$, and 2 is shown.

We have a map $f \colon Z \to Z_0$ given by $f(z_1, \dots, z_k) = (z_1 - z_k, \dots, z_{k-1} - z_k, 0)$. Then f induces a continuous map $f' \colon Z/\mathrm{Euc}(2) \to Z_0/O(2)$. Likewise the inclusion $Z_0 \subset Z$ induces a continuous map $g' \colon Z_0/O(2) \to Z/\mathrm{Euc}(2)$ and these maps are inverses of each other. So 3 is proven.

To see 4, note that if $\mathrm{Tran}(2) \subset \mathrm{Euc}(2)$ is the subgroup of translations, then both Z and Y are the union of $\mathrm{Tran}(2)$ orbits of points of Z_0. So $Y = Z$.

Now suppose that Z is virtually compact. Then by 3 we know that $Z_0/O(2)$ is compact. Suppose $z \notin Z_0$ but z is in the closure of Z_0. Let K be the $O(2)$ orbit of z. For any $\delta > 0$ let U_δ be the set of points in $\mathbb{C}^k$ with distance greater than δ from K. Each U_δ is $O(2)$ invariant, so we get an open cover $\{(U_\delta \cap Z_0)/O(2)\}$ of $Z_0/O(2)$. By compactness of $Z_0/O(2)$ we know that $U_\delta \supset Z_0$ for some $\delta > 0$, contradicting z being in the closure of Z_0. So Z_0 is closed. For any $r > 0$ let B_r be the open ball of radius r around 0. We get an open cover $\{(B_r \cap Z_0)/O(2)\}$ of $Z_0/O(2)$. So $B_r \supset Z_0$ for some r and thus Z_0 is bounded. So Z_0 is compact.

On the other hand, if Z_0 is compact, then $Z_0/O(2)$ is compact, so Z is virtually compact, so 1 is shown. $\square$

We are now ready to prove Theorem 1.2.

Proof. (of Theorem 1.2) We first prove 5-8. So suppose $\mathcal{L}$ is a connected linkage with m fixed vertices and $W \subset \mathcal{V}(\mathcal{L})$. By the Tarski-Seidenberg theorem [6], we know $\mathcal{SC}(\mathcal{L}, W)$ is a semialgebraic set since it is a projection of the algebraic set $\mathcal{C}(\mathcal{L})$.

Note that if $\beta \in \mathrm{Euc}(2)$ and β fixes the images of all fixed vertices of $\mathcal{L}$, and $\varphi \in \mathcal{C}(\mathcal{L})$, then $\beta\varphi \in \mathcal{C}(\mathcal{L})$ also. Consequently $\beta(\mathcal{SC}(\mathcal{L}, W)) \subset \mathcal{SC}(\mathcal{L}, W)$. So if $m = 0$, $\mathcal{SC}(\mathcal{L}, W)$ is invariant under $\mathrm{Euc}(2)$. If $m = 1$, $\mathcal{SC}(\mathcal{L}, W)$ is invariant under the subgroup G of $\mathrm{Euc}(2)$ which fixes the image of the fixed vertex of $\mathcal{L}$. If $m = 2$, $\mathcal{SC}(\mathcal{L}, W)$ is invariant under the subgroup G of $\mathrm{Euc}(2)$ which fixes a line through the images of the two fixed vertices of $\mathcal{L}$. So we have shown everything but compactness.

If $m > 0$, let z_0 be the image of some fixed vertex and let d be the sum of the lengths of all edges of $\mathcal{L}$. Then for each $\varphi \in \mathcal{C}(\mathcal{L})$ and each $v \in \mathcal{V}(\mathcal{L})$ we know that $\varphi(v)$ lies in the ball of radius d around z_0. So $\mathcal{C}(\mathcal{L})$ is bounded. But it is also closed since it is an algebraic set. So $\mathcal{C}(\mathcal{L})$ is compact. Hence $\mathcal{SC}(\mathcal{L}, W)$ is compact since it is the continuous image of $\mathcal{C}(\mathcal{L})$.

If $m = 0$, let $W = \{w_1, \dots, w_k\}$. If $k = 0$ then $\mathcal{SC}(\mathcal{L}, W) = \mathbb{C}^0$ is compact, so assume that $k > 0$. Let $\mathcal{L}'$ be the linkage obtained from $\mathcal{L}$ by fixing the vertex

w_k at 0. Let $Z_0 = \mathcal{SC}(\mathcal{L}', W)$. Note that Z_0 is compact and $O(2)$ invariant by 6. Also $Z_0 = \mathcal{SC}(\mathcal{L}, W) \cap \mathbb{C}^{k-1} \times 0$. So by Lemma 3.4-1, Z is virtually compact so 5 is proven. Note that $\mathcal{SC}(\mathcal{L}, W)$ must be noncompact since it is invariant under Tran(2).

Now we will prove 2, 3, and 4. Note in these cases that Z is compact. After replacing Z by $\beta(Z)$ for some $\beta \in \mathrm{Euc}(2)$, we may assume in case 2 that $G = O(2)$, and may assume in case 3 that Z is invariant under complex conjugation. By Lemma 3.2, we may find a linkage $\mathcal{L}'$ and a $W \subset \mathcal{V}(\mathcal{L}')$ so that $\mathcal{SC}(\mathcal{L}', W) = Z$. Throw away all connected components of $\mathcal{L}'$ which do not contain any vertices in W or any fixed vertices, doing so does not change $\mathcal{SC}(\mathcal{L}', W)$. By adding some isolated fixed vertices to $\mathcal{L}'$ if necessary, we may assume that there is a vertex fixed at 0, another fixed at 1, a third fixed at $\sqrt{-1}$, and a fourth fixed at $-1-\sqrt{-1}$. Adding an isolated fixed vertex to $\mathcal{L}'$ does not change $\mathcal{SC}(\mathcal{L}', W)$.

Let the fixed vertices of $\mathcal{L}'$ be $\{v_1, \ldots, v_k\}$ where v_i is fixed to the point z_i. We may suppose $z_1 = 0$, $z_2 = 1$, $z_3 = \sqrt{-1}$, and $z_4 = -1-\sqrt{-1}$. For each pair i, j with $z_i \neq z_j$ put in an edge $\overline{v_i v_j}$ of length $|z_i - z_j|$, if it is not already there. This will not change $\mathcal{SC}(\mathcal{L}', W')$. Note we did not attempt to add any zero length edges, which would not be allowed.

Let $\mathcal{L}''$ be obtained from $\mathcal{L}'$ by only fixing the vertices v_1, v_2, and v_3. We claim that $\mathcal{SC}(\mathcal{L}', W) = \mathcal{SC}(\mathcal{L}'', W)$. One inclusion $\mathcal{SC}(\mathcal{L}', W) \subset \mathcal{SC}(\mathcal{L}'', W)$ is trivial. So let us see the other inclusion. Pick any $\varphi \in \mathcal{C}(\mathcal{L}'')$. We claim that in fact $\varphi(v_i) = z_i$ for all i. To see this, note first that two different points in $\mathbb{C}$ can't have the same distances from three noncollinear points. Consequently $\varphi'(v_4) = z_4$ since the three edges $\overline{v_i v_4}$, $i = 1, 2, 3$ have lengths $|z_4 - z_i|$, so $|\varphi'(v_4) - z_i| = |\varphi'(v_4) - \varphi'(v_i)| = |z_4 - z_i|$. Then for any $j > 4$, there are edges in $\mathcal{L}$ from v_j to at least three of the v_i, $i \leq 4$, and consequently $\varphi'(v_j) = z_j$ since any three of the z_i, $i \leq 4$ are noncollinear. Consequently, $\varphi \in \mathcal{C}(\mathcal{L}')$. So $\varphi|_W \in \mathcal{SC}(\mathcal{L}', W)$, and we have shown that $\mathcal{SC}(\mathcal{L}', W) = \mathcal{SC}(\mathcal{L}'', W)$.

We claim that $\mathcal{L}''$ is connected. All fixed vertices of $\mathcal{L}$ are in the same connected component since they are all connected by edges. We also threw out any components without points of W. So any other components must have no fixed vertices and must contain points of W. We saw from the proof of 5 above that semiconfiguration spaces of such components are noncompact. Hence by Lemma 3.3, the configuration space of $\mathcal{L}''$ would be noncompact, but it is not. So $\mathcal{L}''$ is connected and so 4 is proven.

Let us now prove 3. Let $\mathcal{L}$ be obtained from $\mathcal{L}''$ by only fixing the vertices v_1 and v_2, and not fixing v_3. We claim that $\mathcal{SC}(\mathcal{L}'', W) = \mathcal{SC}(\mathcal{L}, W)$. Again, one inclusion $\mathcal{SC}(\mathcal{L}'', W) \subset \mathcal{SC}(\mathcal{L}, W)$ is trivial. So let us see the other inclusion. Pick any $\varphi \in \mathcal{C}(\mathcal{L})$. Now $|\varphi(v_3)| = |z_3| = 1$, and $|\varphi(v_2) - \varphi(v_3)| = |z_2 - z_3| = \sqrt{2}$, so the triangles $z_1 z_2 z_3$ and $z_1 z_2 \varphi(v_3)$ are congruent, and hence $\varphi(v_3) = \pm\sqrt{-1}$. If $\varphi(v_3) = \sqrt{-1}$ then $\varphi \in \mathcal{C}(\mathcal{L}'')$ so $\varphi|_W \in \mathcal{SC}(\mathcal{L}'', W)$. If $\varphi(v_3) = -\sqrt{-1}$ then the complex conjugate $\overline{\varphi} \in \mathcal{C}(\mathcal{L}'')$ so $\overline{\varphi}|_W \in \mathcal{SC}(\mathcal{L}'', W) = Z$, but then $\varphi|_W \in \mathcal{SC}(\mathcal{L}'', W)$ since Z is invariant under complex conjugation.

Now let us prove 2. Let $\mathcal{L}$ be obtained from $\mathcal{L}'$ by only fixing the vertex v_1 to 0, and not fixing any of the other vertices of $\mathcal{L}'$. We claim that $\mathcal{SC}(\mathcal{L}'', W) = \mathcal{SC}(\mathcal{L}, W)$. One inclusion $\mathcal{SC}(\mathcal{L}'', W) \subset \mathcal{SC}(\mathcal{L}, W)$ is trivial. So let us see the other inclusion. Pick any $\varphi \in \mathcal{C}(\mathcal{L})$. Now $|\varphi(v_2)| = |z_2| = 1$, $|\varphi(v_3)| = |z_3| = 1$, and $|\varphi(v_2) - \varphi(v_3)| = |z_2 - z_3| = \sqrt{2}$, so there is a $\beta \in O(2)$ so that $\beta(\varphi(v_2)) = z_2 = 1$ and

$\beta(\varphi(v_3)) = z_3 = \sqrt{-1}$. For convenience, let $\varphi' = \beta \circ \varphi$. Note that $\varphi' \in \mathcal{C}(\mathcal{L}'')$. So $\varphi'|_W \in \mathcal{SC}(\mathcal{L}'', W) = Z$. By $O(2)$ invariance of Z, we know that $\beta^{-1} \circ \varphi'|_W \in Z$ also. But $\beta^{-1} \circ \varphi'|_W = \varphi|_W$, so $\varphi|_W \in Z$. So we have shown that $\mathcal{SC}(\mathcal{L}, W) \subset \mathcal{SC}(\mathcal{L}'', W)$, and hence $\mathcal{SC}(\mathcal{L}, W) = Z$.

Finally, let us prove 1. Let $Z_0 = Z \cap \mathbb{C}^{n-1} \times 0$. By Lemma 3.4-1 and 2, Z_0 is compact and $O(2)$ invariant. Hence by 2 there is a $\mathcal{L}'$ and W so that $Z_0 = \mathcal{SC}(\mathcal{L}', W)$ and $\mathcal{L}'$ has only one fixed vertex, fixed at 0. Form $\mathcal{L}$ from $\mathcal{L}'$ by unfixing this vertex. Then $\mathcal{SC}(\mathcal{L}, W) \cap \mathbb{C}^{n-1} \times 0 = \mathcal{SC}(\mathcal{L}', W) = Z_0$. So $\mathcal{SC}(\mathcal{L}, W) = Z$ by Lemma 3.4-4. □

We can now prove Theorem 1.1.

Proof. (of Theorem 1.1) By Lemma 3.3, it suffices to prove the equivalence of 2(a) and 2(b). The equivalence of 1(a) and 1(b) will follow by applying the equivalence of 2(a) and 2(b) to each connected component of $\mathcal{L}$ which contains vertices of W.

We know from Theorem 1.2-8 and 5 that 2(a) implies 2(b), and along with Lemma 3.3 we get all the extra bits of information.

So it remains to prove 2(b) implies 2(a). If X is compact, this follows from Theorem 1.2-4. Suppose now X is not compact, so X is invariant under the action of Euc(2) with compact quotient. Then 2(a) follows from Theorem 1.2-1. □

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